# Landmark-Constrained Elastic Shape Analysis of Planar Curves


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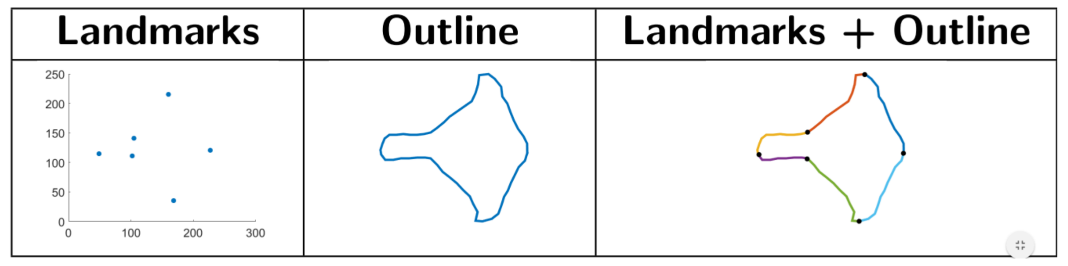
## Introduction

Statistical shape analysis: application of statistical methods to shapes of objects

- Shape: property of object which is invariant to rotation, translation, and scaling of object
- Types of shape representations:
  - Landmark-based – ignores full outline
    - Landmarks – finite set of important points on shape
  - Functional representations – ignores landmark points
    - Requires additional invariance to re-parameterization

## Aims

- To combine benefits of both types of representations without ignoring available information
- To illustrate differences (and improvements) in the performance of statistical techniques due to this improved shape representation




## Methods

### Elastic Shape Analysis Framework

**Goal:** Develop a metric on the space of shapes.

Curve: $\beta : D \to \mathbb{R}^2$ $D$ = curve domain (open: [0,1], closed: $\mathbb{S}^1$) (absolutely continuous curve)

*Unconstrained re-parameterization group:*

$$\Gamma = \{\gamma : [0,1] \to [0,1] | \gamma(0) = 0, \gamma(1) = 1, 0 < \dot{\gamma} < \infty\}$$

➢ Re-parameterized curve: $\beta \circ \gamma$

"Simple" metric: $||\beta_1 - \beta_2|| = \sqrt{\int_0^1 |\beta_1(t) - \beta_2(t)|^2 \, dt}$

Problem: Not invariant to re-parameterizations: $||\beta_1 - \beta_2|| \neq ||\beta_1 \circ \gamma - \beta_2 \circ \gamma||$

Solution: Define *square root velocity function* (SRVF): $q(t) = \frac{\dot{\beta}(t)}{\sqrt{|\dot{\beta}(t)|}}$

- *q* includes instantaneous velocity info for $\beta$
- Automatically invariant to translations
- Re-scaling $\beta$ corresponds to SRVF lying on unit Hilbert sphere:

$$\mathcal{C} = \{q : [0,1] \to \mathbb{R}^2 | \int_0^1 |q(t)|^2 dt = 1\}$$

- $\mathbb{L}^2$ metric on $\mathcal{C}$ is invariant to re-parameterizations (equivalent to elastic metric)

Geodesic distance: $\theta = d_{\mathcal{C}}(q_1, q_2) = \cos^{-1}(\langle q_1, q_2 \rangle)$

Geodesic path: $\alpha_{q_1,q_2}(\tau) = \frac{1}{\sin(\theta)}(\sin((1-\tau)\theta)q_1 + \sin(\tau\theta)q_2), \tau \in [0,1]$

### Introducing Landmarks

Suppose in addition to full curve $\beta$, we are given *k* discrete landmark points $\{\beta(t_1), \ldots, \beta(t_k)\}$.

- *Landmark-constrained re-parameterization group:* $\Gamma_0 = \{\gamma : [0,1] \to [0,1] | \gamma(0) = 0, \gamma(1) = 1, 0 < \dot{\gamma} < \infty, \gamma(t_i) = t_i, i = 1, \ldots, k\}$
- Space of all 2 x 2 rotation matrices: $SO(2)$

Landmark-constrained shape space: $[q] = \{O(q \circ \gamma)\sqrt{\dot{\gamma}} | O \in SO(2), \gamma \in \Gamma_0\}$

$$\mathcal{S} = \mathcal{C}/(SO(2) \times \Gamma_0)$$

[If also interested in size, can similarly form landmark-constrained size-and-shape space by not re-scaling curve to unit length initially]

### Comparison of Shapes

Geodesic distance on $\mathcal{S}$: $d_{\mathcal{S}}([q_1],[q_2]) = \min_{O \in SO(2), \gamma \in \Gamma_0} d_{\mathcal{C}}(q_1, O(q_2 \circ \gamma)\sqrt{\dot{\gamma}})$

$$= \cos^{-1}\left(\left\langle q_1, O^*(q_2 \circ \gamma^*)\sqrt{\dot{\gamma}^*} \right\rangle\right)$$

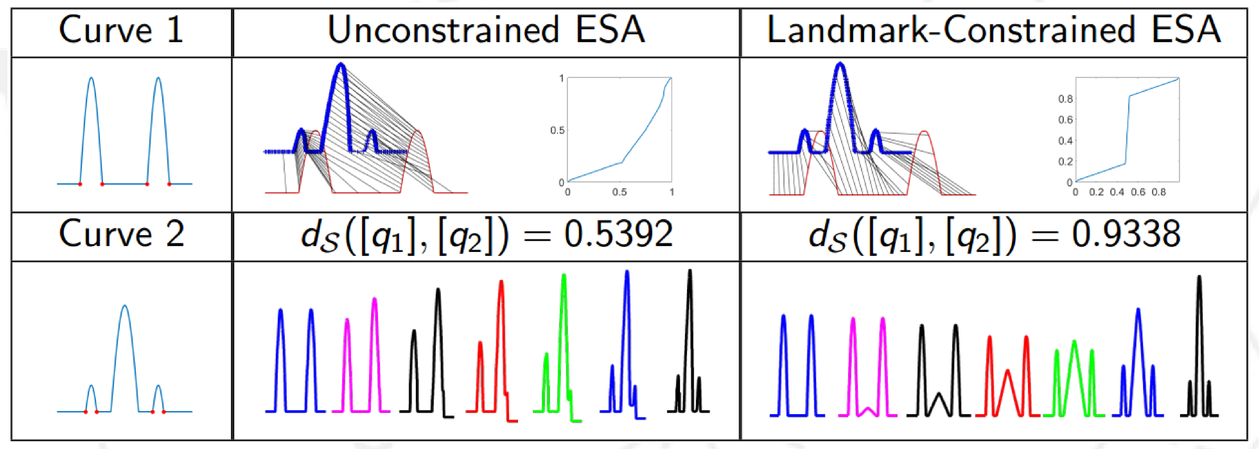
We fix $q_1$ and find optimal rotation and re-parameterization of $q_2$ to $q_1$

- Optimal rotation – Procrustes analysis
- Optimal re-parameterization – dynamic programming (split into many sub-problems based on landmarks) OR gradient-descent

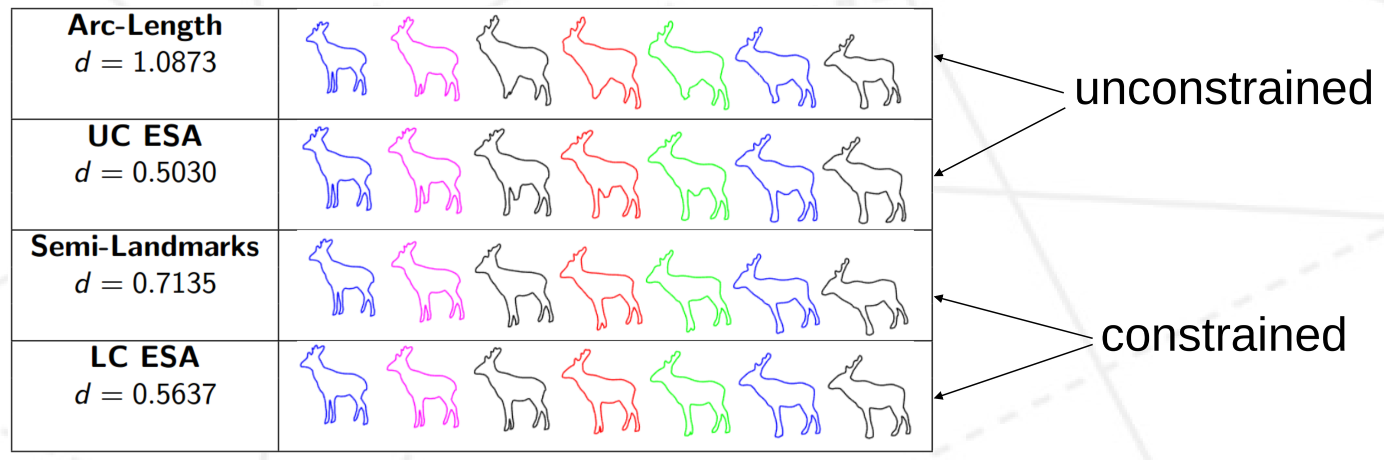
Geodesic path on $\mathcal{S}$: $\alpha_{q_1, O*(q_2 \circ \gamma^*)\sqrt{\dot{\gamma}^*}}$

## Results

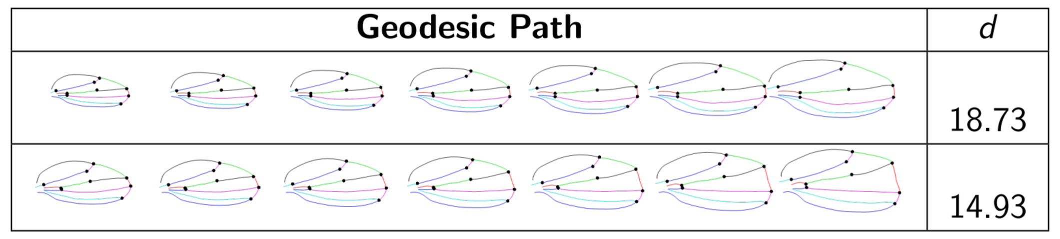
### Simulated Example

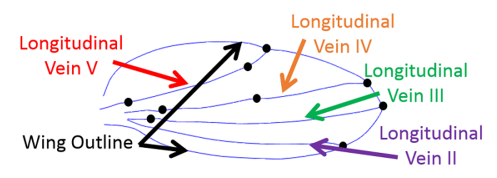



### Comparison to Previous Methods via Geodesic Paths



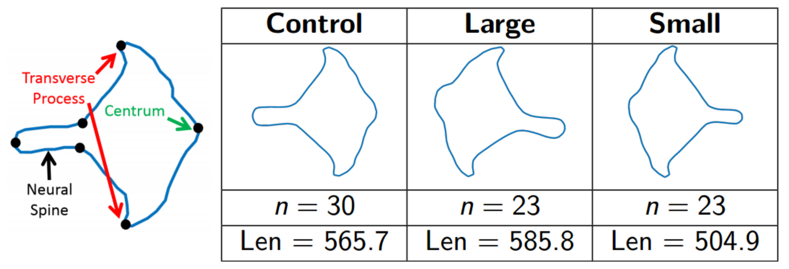

**Fly Wings –** not represented by single curve but can use LC ESA on product space



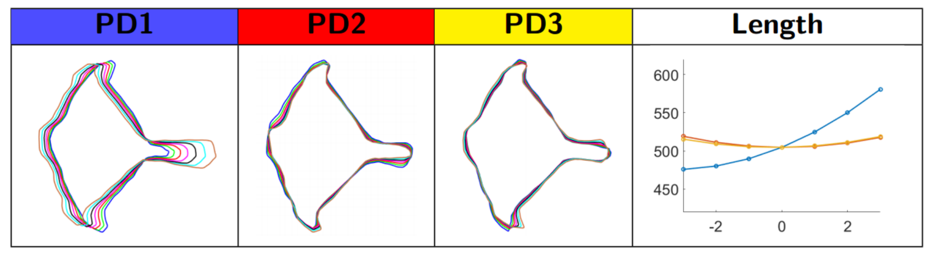




### Statistical Methods

- Karcher mean: $[\bar{q}] = \operatorname{argmin}_{[q] \in \mathcal{S}} \sum_{i=1}^n d_{\mathcal{S}}([q],[q_i])^2$ (requires gradient-descent)




- Tangent PCA: allows for visualization of dominant modes of variation



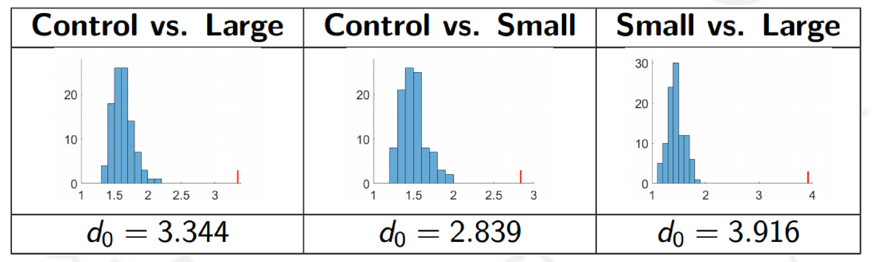

- Classification: using leave-one-out nearest average approach (classify to closest group mean)

number of misclassifications per group

| | Landmark | UC ESA | LC ESA |
|---|---|---|---|
| **Control** | 5 | 5 | **2** |
| **Large** | 5 | **2** | 3 |
| **Small** | 2 | 2 | **1** |
| Total | 12 | 9 | **6** |

- Two-Sample Hypothesis Tests: used to test equality of mean shapes (permutation test based on dist. of permuted group mean distances)

$$H_0 : [\mu_1] = [\mu_2] \text{ vs. } H_1 : [\mu_1] \neq [\mu_2]$$




$d_0 = d_{\mathcal{S}}([\bar{q}_1],[\bar{q}_2])$

## Conclusions

- More natural geodesic paths when guided by landmarks
- Many common statistical techniques used for shape analysis can be extended with the landmark-constrained framework; improvement can be found due to the inclusion of both overall shape outline and important landmark selection

## Bibliography


J. Strait, S. Kurtek, E. Bartha, S. MacEachern, "Landmark-Constrained Elastic Shape Analysis of Planar Curves," *Journal of the American Statistical Association*, Accepted for Publication, 2016.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